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COMMENT

A finite-size scaling study of the 4D Ising model

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Abstract. We study the finite-size scaling behaviour of the susceptibility and of the second derivative of the susceptibility with respect to the magnetic field at the critical point for the four-dimensional Ising model with zero magnetic field using Monte Carlo techniques. The finite-size scaling exponents are found to be in good agreement with the form predicted by mean-field theory critical exponents.

Finite-size scaling techniques (for a good introduction and references see [1]) have become a very powerful method of analysing the critical behaviour of statistical mechanical systems. This is particularly true when they are combined with results obtained by analytical methods. In this study we investigate the finite scaling properties of the magnetic susceptibility and its second derivative with respect to the magnetic field at the critical point for the four-dimensional Ising model. We will use the known value of the critical temperature obtained from high-temperature series expansions to fine tune our Monte Carlo measurements, and check the theoretical expectations for the finite-size scaling exponents and the possibility of using them to extract information about the critical properties of the system. Because the critical exponent γ is exactly known in this model [2], only two more critical exponents need to be measured. If hyperscaling is satisfied, this can be done in an economical way, as explained below, by analysing the finite-size behaviour of the magnetic susceptibility. If hyperscaling is not satisfied, then all the critical exponents except ν and η can be obtained in that manner.

The four-dimensional Ising model is a system of special interest. It is generally accepted that its critical properties are given by mean-field theory, with possible logarithmic corrections. This is what renormalisation group calculations [3] suggest, giving also detailed predictions about the logarithmic terms modifying the power law singularities. In four dimensions, and only in four dimensions, the mean-field theory exponents satisfy the hyperscaling relation, whose validity is implicitly assumed in the renormalisation group approach:

$$\omega^* \equiv \frac{-2\Delta + \mathrm{d}\nu + \gamma}{\nu} = 0 \qquad (1)$$

where Δ, ν and γ are the usual critical exponents. In general, one has the inequality $\omega^* \geq 0$ [4]. This exponent ω^* is the relevant quantity to discuss the issue of the triviality of the continuum field theories obtained at the critical point, because the renormalised coupling constant behaves when $T \to T_c$ as:

$$g_R \equiv \frac{\chi^{(2)}}{\xi^d \chi^2} \sim t^{\nu\omega^*} \qquad t \equiv \frac{|T - T_c|}{T_c} \qquad (2)$$

where χ and $\chi^{(2)}$ are respectively the susceptibility and the second derivative of the susceptibility at zero magnetic field. A violation of the hyperscaling relation (1) will automatically imply a trivial (Gaussian) continuum theory and indeed this is what happens in $d > 4$. The common belief in four dimensions, where mean-field theory exponents satisfy (1), is that logarithmic corrections to equation (2) are responsible for the (almost proven) triviality of the continuum theory. This mild way of vanishing g_R when $T \to T_c$ is in contradiction with some Monte Carlo calculations [5], which actually *see* g_R go towards zero very fast as the temperature approaches its critical value.

Baker and Kincaid [6] have analysed high-temperature series expansions for different lattice types using integral and Padé approximants, and have obtained the following value of ω^*:

$$\nu\omega^* = 0.302 \pm 0.038 \tag{3}$$

which implies strong triviality, and is in conflict with mean-field theory exponents. Baker suggested that the renormalisation group calculations may not apply to the four-dimensional Ising model. On the other hand, Gaunt and co-workers [7, 8] *assuming* the validity of the mean-field exponents and thus of equation (1), have found consistency between the high-temperature series expansions and the logarithmic corrections predicted by the renormalisation group. Aizenman and Graham [2] have rigorously proven that there are at most logarithmic corrections to the mean-field theory prediction for the behaviour of the magnetic susceptibility. Furthermore, they showed that if these corrections are present then the continuum limit is trivial, although the way in which g_R goes to zero is not specified, only bounded by a logarithmic term, and a violation of hyperscaling is not ruled out.

It is, therefore, of interest to see if one can obtain information about the critical behaviour of the theory using numerical methods, in order to support or reject the standard picture. In this comment we study the finite-size scaling [1] behaviour of the susceptibility χ and the second derivative of the susceptibility $\chi^{(2)}$ at zero magnetic field at the critical temperature, using Monte Carlo techniques in hyper-cubical lattices of linear size L with periodic boundary conditions. For the critical temperature we use the known value obtained in series expansions [7]:

$$\beta_c = \frac{J}{kT_c} = 0.149\,65 \pm 0.000\,05. \tag{4}$$

The accuracy of this value is enough for our simulation.

The values of $\chi(T_c, L)$ and $\chi^{(2)}(T_c, L)$ diverge when the size of the lattice go to infinity as [1]:

$$\chi(T_c, L) \sim L^{\lambda_1}. \tag{5a}$$

$$\chi^{(2)}(T_c, L) \sim L^{\lambda_2}. \tag{5b}$$

Under certain hypotheses which do not require the validity of equation (1), Binder *et al* [9] have found the following relation between the finite-size exponents λ_1 and λ_2 and between them and the critical exponents γ and β:

$$\lambda_2 = 2\lambda_1 + d \tag{6a}$$

$$\lambda_1 = 2d\frac{\gamma + \beta}{\gamma + 2\beta} - d. \tag{6b}$$

If equation (1) is satisfied, then the finite-size exponents λ_1 and λ_2 take the more familiar forms [1]:

$$\lambda_1 = \gamma/\nu \tag{7a}$$

$$\lambda_2 = (2\Delta + \gamma)/\nu. \tag{7b}$$

The 'mean-field theory' values for λ_1 and λ_2 are 2 and 8 respectively. If the hyperscaling relation (1) fails, then at least two critical exponents must differ from their mean-field theory values. If λ_1 and λ_2 still take the values 2 and 8, then the exponent beta must also take its mean-field theory value and only ν (and $\eta = 2 - \gamma/\nu$) can deviate from their mean-field theory prediction. We will check numerically the validity equation of (6*a*). Then, using (6*b*) and (7), together with the known exact result of $\gamma = 1$ [2], we can estimate bounds to the deviation of the critical exponents from their mean-field theory values.

For our numerical simulations, we have used hypercubical lattices with linear sizes L from 3 to 10. A standard heat bath method was used to update the spins. For each lattice size, we measured χ and $\chi^{(2)}$ by generating several Markov chains of configurations. Each of them starting with a random initial spin configuration and consisting of 5×10^3–10^4 Monte Carlo sweeps for thermalisation followed by 10^4–10^5 sweeps for measurement, each two measurements separated by 20–200 sweeps. This allowed a better understanding of the statistical errors involved in the measurement.

Specifically, our definitions of χ and $\chi^{(2)}$ are:

$$\chi = L^4\langle m^2\rangle \tag{8a}$$

$$\chi^{(2)} = -L^{12}(\langle m^4\rangle - 3\langle m^2\rangle\langle m^2\rangle) \tag{8b}$$

where m represents the magnetisation *per site*. These definitions coincide up to factors of β_c with the standard definitions. The minus sign in front of the left-hand side in (8*b*) will assure a positive value for $\chi^{(2)}$. The results of our measurements are given in table 1, and are plotted in figures 1 and 2. The main source of the errors is statistical in nature. From these figures it is apparent that the asymptotic region (5*a*, *b*) is reached very quickly (within the accuracy of the measurements), even for lattices as small as $L = 5$. From the values of χ and $\chi^{(2)}$ in table 1, our best estimates of the values of λ_1 and λ_2 are:

$$\lambda_1 = 1.96 \pm 0.06 \tag{8c}$$

$$\lambda_2 = 7.85 \pm 0.25 \tag{8d}$$

Table 1. Measured values of χ and $\chi^{(2)}$ for different values of the size of the lattice.

L	$\chi(T_c)$	$\chi^{(2)}(T_c)$ ($\times 10^6$)
3	16.310 ± 0.006	0.02546 ± 0.00005
4	33.22 ± 0.03	0.330 ± 0.002
5	55.8 ± 0.3	2.23 ± 0.04
6	84.8 ± 0.8	10.6 ± 0.2
7	120.0 ± 1.7	39.0 ± 1.3
8	159.7 ± 2.2	116.0 ± 6.1
9	204 ± 6	295 ± 25
10	258 ± 8	709 ± 40

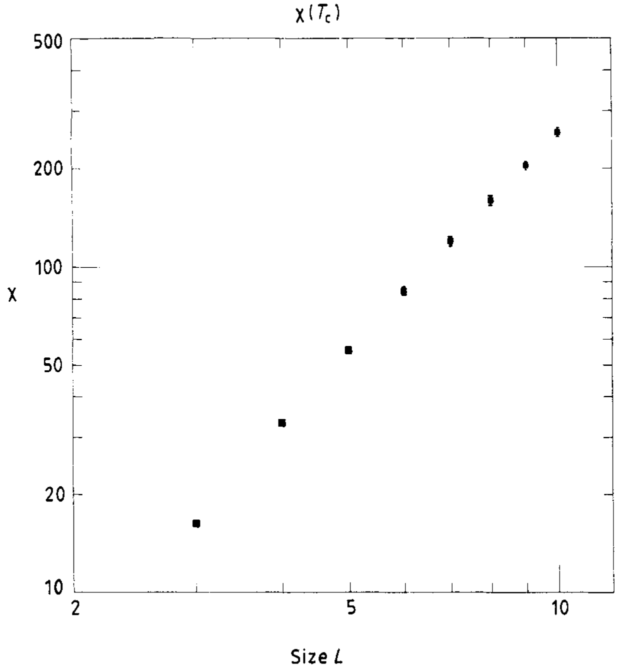

Figure 1. Magnetic susceptibility χ at zero magnetic field defined in equation (8a) at the critical temperature as a function of the lattice size L.

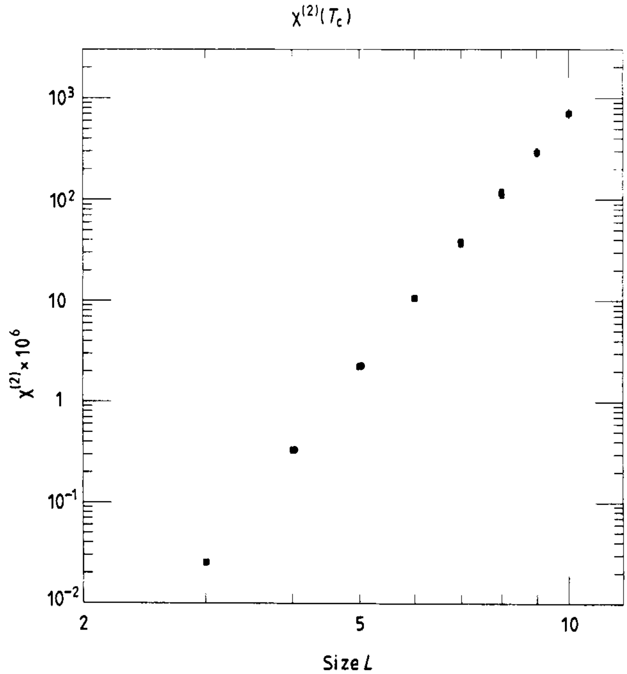

Figure 2. Second derivative of the magnetic susceptibility $\chi^{(2)}$ with respect to the magnetic field at zero magnetic field defined in equation (8b) at the critical temperature as a function of the lattice size L.

which are in very good agreement with the prediction of equation (6*a*) [9]. They are also consistent with their 'mean-field theory' values of 2 and 8. The errors quoted in equation (8) are estimated errors and not rigorous bounds. Using equation (6*b*) together with the exact value $\gamma = 1$ and the scaling relations, which do not assume the validity of hyperscaling, we obtain for the critical exponents α, β, δ and Δ:

$$\beta = 0.52 \pm 0.03\ [\tfrac{1}{2}] \tag{9a}$$

$$\alpha = 2 - 2\beta - \gamma = 0.04 \pm 0.06\ [0] \tag{9b}$$

$$\delta = 1 + \gamma/\beta = 2.92 \pm 0.12\ [3] \tag{9c}$$

$$\Delta = \beta + \gamma = 1.52 \pm 0.03\ [\tfrac{3}{2}] \tag{9d}$$

where inside the brackets we have indicated their mean-field theory values. If we further assume the validity of equation (1) then we find for ν and η:

$$\nu = 0.510 \pm 0.016\ [\tfrac{1}{2}] \tag{10a}$$

$$\eta = 2 - \gamma/\nu = 0.04 \pm 0.06\ [0] \tag{10b}$$

also consistent with their mean-field values. The Monte Carlo data for the specific heat C_V at the critical temperature is presented in figure 3. It is also consistent with α being zero or very small, although it would not be correct to extract more precise conclusions from it than those obtained in (9*b*). The fact that C_v diverges very slowly with the size of the lattice means that one will need data from bigger lattices and more statistical accuracy to get a better estimate or bound for α from a direct measurement

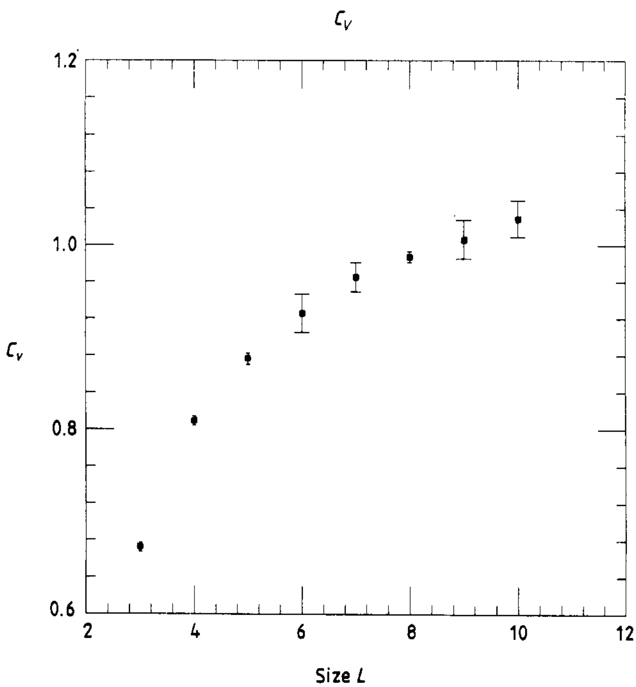

Figure 3. Specific heat at the critical temperature as a function of the lattice size L.

of C_v. However, if we assume a logarithmic divergence of $C_v(T_c)$ when the size of the lattice goes to infinity (i.e. assuming $\alpha = 0$):

$$C_v(T_c, L) \sim \log^p(L) \tag{11}$$

then our data are consistent with a value of $p \simeq 0.3$–0.45.

In summary, we have found for the finite-size scaling behaviour of the susceptibility and its second derivative with respect to the magnetic field, at zero magnetic field, very good agreement with the expected scaling (5) and (6a), with critical exponents consistent with those given by mean-field theory. This is in agreement with the renormalisation group results for the critical exponents, which in turn imply the hyperscaling relation (1). We have also found a very fast approach to the large-L limit for these quantities, even for sizes of the lattice relatively small. We thus believe that it will be possible to improve the accuracy of the results presented here by concentrating on increasing the statistics of the measurements in small lattices (of the order 10).

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